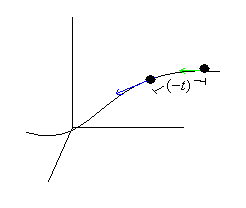
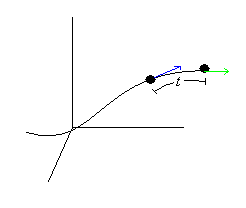
**Time Reversal Operator**

Now let’s consider time reversal symmetry. The time reversal operator is kind of like the parity operator, but applied to time, i.e., it maps t to –t. Actually it is better to think of it in a different way. In a sense, the time reversal operator should be called the motion reversal operator. This is because, classically, what the time reversal operator does is to reverse the motion of the particle, and this is accomplished simply by reversing its velocity, or equivalently, its momentum **p**, and then progressing time forward. If the system possesses time-reversal symmetry, then the particle will trace out its prior path up to that point. If the system doesn’t possess TRS then it will evolve elsewhere in space. This is depicted in the graph below, for a system that does possess TRS.



Classically then, we would say that a system possesses TRS if **r**(t) being a solution to the equations → **r**(-t) is also a solution to the equations, because **r**(-t) would be the solution obtained upon negating **p**. For example, let’s consider the Lorentz force law:



For fixed **E** and **B**, this equation doesn’t possess TRS, since **r**(-t) isn’t a solution if **r**(t) is, since plugging in **r**(-t) yields.



On the other hand, if one recognizes that **B** would actually reverse direction under TR (since **B** is made up of currents which would reverse direction if time were reversed) then **B** → – **B** and we would get TRS. So the system *as a whole* would possess TRS.

Anyway, let’s try to determine how to represent the time reversal operator in quantum mechanics, and determine what its action on wavevectors and operators is. ΘT So let Θ be the time-reversal operator (we choose Θ because it and time both start with the same letter). ΘNow with the other symmetry operators, we knew their action on a given ket, and so we could from that deduce their representation in the HS. Θ is a little harder to figure out since its action on a ket is rather vague – it reverses its motion. Nonetheless we can tease out a few properties. First we might postulate that since Θ is the motion reversal operator, then we’d have:



and we can take this to be a defining property – almost (recall the parity operator also does this and so if this were the defining property of Θ, then Θ and P would be the same). Turns out Θ is not a linear operator, and so knowing its action on base kets of the Hilbert Space is not enough to know its operation on every single ket. For instance, consider a wavefunction |ψ>, and let Θ|ψ> be the time reversed wavefunction. If the Hamiltonian has TRS, then we must have that evolving a ket |ψ> through time δt and then motion reversing is the same as first motion reversing it, and then evolving it backwards in time δt (think of |ψ> as a particle’s velocity vector). In other words we should have:



Now observe that if we were to assume that we can pass the iδt through Θ and cancel them out, that we would have {H,Θ}= 0, but this cannot be since this would imply Θ-1HΘ = -H, and so H wouldn’t be invariant w/r to Θ. Rather it must be the case that passing the *i* through Θ complex conjugates it. Then we’d have:



as required. Thus we see that Θ must be an *anti-linear* operator, meaning that:



This property has rather grave consequences for Θ w/r to the Hilbert space. Only linear operators can be *represented* on the Hilbert space, and so this means that Θ cannot be. So we cannot say something like,



We can always determine what Θ’s action on a ket is; we just can’t represent it as a matrix which is necessarily a linear operator. And this is why we cannot use simply the first relation to determine the action of Θ on all other objects in the HS. These two properties imply that Θ is a so-called anti-unitary operator, which in effect means it can be represented as the product of a unitary and a complex conjugating operator, with the following defining properties.



We’re not going to really break Θ up into its two parts ever, in practical examples. Only breaking up its two properties pedagogically. One more comment. |p> is not a special ket per se´. We can substitute |p> with any other basis ket, as long as we also update the U|p> = |-p> equation with its equivalent on the other basis ket. K’s behavior remains unchanged.

**Example**

Given the above properties, let’s prove the following.



So this follows from:



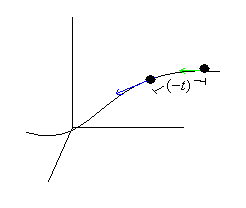
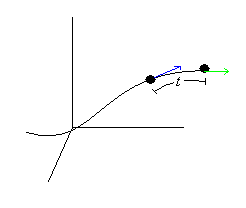
and so,



So there we go.

**Effect of Θ on |ψ(t)>**

So we said above that classically, if H possesses TRS, then if r(t) is a solution to the equations of motion, r(-t) is as well. Let’s work out the quantum mechanical analog. So if H possesses TRS, then both |ψ(t)> and Θ|ψ(-t)> is a solution to the Schrodinger equation. This should be intuited upon a realization of what this represents.



|ψ(t)> is represented by the trajectory on the left. |ψ(-t)> is the left trajectory reversed, but with the arrows going opposite the ‘velocity’. Θ reverses the ‘arrows’ (e.g. momentum, etc.) and makes this a physically meaningful trajectory, i.e., makes the motion align with the direction of traversal. So…



So there. What does this reversed wavevector look like in position space? The evolved time-reversed wavevector is defined via |ψ(t)>TR = |ψ(-t)> (remember reverses the direction of motion, but doesn’t evolve in time per se´, so we need to insert the (-t) argument ourselves).



and then projecting against <**r**| we get:



and so we get, finally,



**Θ’s action on other basis kets**

Alright, now let’s see what effect Θ has on other kets and operators. We’ll have:



The first follows from:



We obviously have that:



As for angular momentum eigenstates we have:



That property of the spherical harmonics follows from the fact that:



For comparison’s sake, using the lowering operator we can write:



and so we have:



The last relationship is taken as more or less a definition of how Θ acts on spinors, by analogy with regular angular momentum states.

**Θ-1…Θ action on operators**

We would also like to examine the effect of the TR operator on other operators as well. Since Θ is not unitary, Θ-1AΘ will be different than Θ†AΘ, alas. Starting with the first set, we have:



These all reflect what we’d think the time-reversal operator would do. These can be proved as follows:



and so then we have:



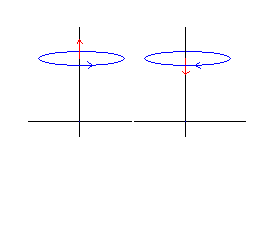
and so our relationship is proved. We can do similarly for the **p**. For the spin operator (and angular momentum operator) it would work like this:



and so, at least for the z-component, we get:



and so we have it proved. These equations should make sense on classical grounds…Time (motion) reversing **r** at a given instant doesn’t change **r**. Time (motion) reversing **p** at a given instant would reverse the orientation of **p** and so it should give –**p**. And time reversing the orbital angular momentum will make it have the opposite direction angular momentum (but same magnitude) as shown below:



**Θ†…Θ action on operators**

Now Θ isn’t unitary, and so Θ†AΘ would give a different result than Θ-1AΘ. Let’s work out what these would be. We’ll find:



These also kind of reflect our expectations. The first follows from the fact that:



And since the expectation holds for all kets, it is functionally an operator identity. We can do the momentum operator as well,



and so the second follows, as well as the third. For the last we’ll note the following operations,



Now observe that if s is a half-integer, then m = ±1/2, ±3/2, etc., then 2m = odd, and so this works out to:



and so the identity holds. On the other hand, if s were an integer, then so would m be, and 2m would be even. Therefore we’d still have an overall (-) sign out front. And so our formula works out – for the angular momentum states too. So actually that last formula is quite general and we could write **J** instead of **S**.

**Kramer’s Degeneracy**

The reasoning behind that last identity leads to something called Kramers degeneracy. Observe that for a ket |jm> where j is half-integer we have:



On the other hand, if j were an integer, then we’d have:



Now suppose that H possesses TRS, i.e., [H,Θ] = 0. Then it must be the case that if j is half-integer, each eigenvalue must be at least two-fold degenerate. For suppose otherwise. Then we have:



And so if the eigenstate isn’t degenerate, then this implies that:



Now |c|2 is positive-definite and so cannot be equal to (-1). Therefore the eigenvalue cannot be non-degenerate. So there we have:



**Example**

Examine  on the basis of TRS again,



Don’t say much it doesn’t seem.

**Example**

Show that introducing **B**-field violates the TRS of Hamiltonian





So it does change the Hamiltonian.

**Example**

Say ΘA = AΘ. Then what is does Θf(A) equal? Well, let’s expand in Taylor series,



But now ΘAΘ-1 = A, since ΘA = AΘ → ΘAΘ-1 = AΘΘ-1. So this is:



**Example**

Or say ΘA = -AΘ. Then what is does Θf(A) equal? Well, let’s expand in Taylor series,



But now ΘAΘ-1 = -A, this is:



**Example**

Can give a more explicit representation of Θ for spin ½ systems. Consider a spin up eigenket in the **n** direction, and we’ll apply the time-reversal operator to it, after using the rotation operators to express it in terms of a spin-up eigenket in the z direction:



But then also have:



where in the first line we basically undo all the rotations to take |s,1/2>n to |s,1/2>, and then do a π rotation about y-axis to get |s,-1/2>. Substituting this into the former, we have:



Well, recall we found in the rotations file, that:



And these two guys commute,



So we can say,



which tells us that:



Well, actually, this isn’t quite true. Since the complex conjugation operator doesn’t have any effect on base kets, we’ve inadvertently only worked out the unitary operator part of Θ, i.e., U. And the full Θ is:



We could also kind of back up and note that:



So that we can say,

